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INFLUENCE OF AMPLITUDE OF VIBRATIONS ON LOSS FACTORS OF LAMINATED COMPOSITE BEAMS AND PLATES

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1. INTRODUCTION

The introduction of structural components made of advanced composite materials such as laminated beams and plates has assumed much importance in the design of aerospace structures; as a consequence, a large number of publications on the subject has appeared in recent years. Further, in order to control the resonant amplitudes of vibration and thus extending service life of such structures under periodic load/impact, the damping in the composite materials plays an important role. For fibre reinforced composites, damping value is higher, in general, compared to that of metallic structures, and it has been brought out in the literature that the system loss factors depend on fibre and resin types, ply-angle, lay-up, and thickness and aspect ratios.

Research on the damping analysis of laminated fibre reinforced composites is not so extensive as that on the undamped free and forced vibration analysis. Gibson and Plunkett [1], and Gibson [2] reviewed experimental and analytical efforts to characterize the damping properties of fibre reinforced materials. Important contributions are cited here. The analysis of vibration and damping of fibre reinforced composite plates has been carried out by Alam and Asnani [3], Malhotra *et al*. [4], and Koo and Lee [5]. Alam and Asnani [3] employed a solution in the form of a series summation and the finite element procedure was adopted by Malhotra *et al*. [4], and Koo and Lee [5]. They are all based on linear dynamic analysis using the correspondence principle of linear visoelasticity. Even though a large amount of work has been carried out on the non-linear dynamics of continuum media, investigation into the damping behavior of the fibre reinforced composite beams/plates using non-linear dynamic analysis has not received much attention in the literature. Such studies are essential for the development of structural design strategies.

In this paper, an attempt is made through non-linear dynamic analysis, to study the influence of amplitude of vibrations on the damping behavior of reinforced composite laminates using the finite elements developed recently based on shear deformation theory, as outlined in the work of Touratier [6], Ganapathi *et al*. [7],

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and Beakou and Touratier [8]. The complex eigenvalue problem based on complex elastic moduli is formulated and the geometric non-linearity due to moderately large deformation has been included based on von Karman's theory. The non-linear governing equations obtained here are solved using a direct iteration technique. The non-linear frequency values and in turn, system loss factors, are obtained for various values of amplitudes.

2. FORMULATION

The displacement field of rectangular shear deformable plates can be expressed as

$$u^{(k)}(x, y, z, t) = u(x, y, t) - z\,\partial w/\partial x + [f_1(z) + g_1^{(k)}(z)]\gamma_1^o + g_2^{(k)}(z)\gamma_2^o,$$

$$v^{(k)}(x, y, z, t) = v(x, y, t) - z\,\partial w/\partial y + g_3^{(k)}(z)\gamma_1^o + [f_2(z) + g_4^{(k)}(z)]\gamma_2^o,$$

$$w^{(k)}(x, y, z, t) = w(x, y, t), \tag{1}$$

where (u^k, v^k and w^k) are the displacements in the kth layer along the (x, y, z) co-ordinates, and (u, v, w) and (θ_x, θ_y) are the displacements and rotations of a point on the middle surfaces, respectively, and $\gamma_1^o(=\partial w/\partial x + \theta_x)$, $\gamma_2^o(=\partial w/\partial y + \theta_y)$ are the shear strains at $z = 0$, and t is the time. Finally, the functions involved in equation (1) for defining the kinematics are as follows:

$$f_1(z) = h/\pi \sin(\pi z/h) - h/\pi b_{55} \cos(\pi z/h),$$

$$f_2(z) = h/\pi \sin(\pi z/h) - h/\pi b_{44} \cos(\pi z/h),$$

$$g_i^{(k)} = a_i^{(k)} z + d_i^{(k)}, \qquad i = 1, 2, 3, 4; \qquad k = 1, 2, 3, \ldots, N, \tag{2}$$

where N is the number of layers of the multi-layered structure, π is equal to 3·141592, h is the total thickness of the plate and b_{44}, b_{55}, $a_i^{(k)}$, $d_i^{(k)}$ are coefficients to be determined from contact conditions for displacements and stresses between the layers and from the boundary conditions on the top and bottom surfaces of the plate. The details of the derivations of these coefficients can be found from references [7, 8].

It can be noted here [6–8] that, unlike in the other theories, trigonometric functions are introduced in the present displacement fields to approximate the shear distribution through the thickness of the plate. The plate model requires only five generalized displacement fields, and no shear correction factors, and it satisfies the interlayer continuity requirements on displacements and stresses.

The strain–displacement relations consisting of linear strain components, which allow cosine variation of transverse shear strain with vanishing shear stresses at the top and bottom of the plate, and non-linear strain components based on von Karman's theory, can be written as follows:

$$\varepsilon_x = u^x - zw^{xx} + [f_1(z) + g_1^{(k)}(z)]\gamma_1^{ox} + g_2^{(k)}(z)\gamma_2^{ox} + (1/2)(w^x)^2,$$

$$\varepsilon_y = v^y - zw^{yy} + g_3^{(k)}(z)\gamma_1^{oy} + [f_2(z) + g_4^{(k)}(z)]\gamma_2^{oy} + (1/2)(w^y)^2,$$

$$\gamma_{xy} = u^y + v^x - 2zw^{xy} + [f_1(z) + g_1^{(k)}(z)]\gamma_1^{oy} + g_2^{(k)}(z)\gamma_2^{oy}$$
$$+ g_3^{(k)}(z)\gamma_1^{ox} + [f_2(z) + g_4^{(k)}(z)]\gamma_2^{ox} + (w^x w^y),$$
$$\gamma_{xz} = [f_1^z(z) + g_1^{(k)z}(z)]\gamma_1^o + g_2^{(k)z}(z)\gamma_2^o,$$
$$\gamma_{yz} = g_3^{(k)z}(z)\gamma_1^o + [f_2^z(z) + g_4^{(k)z}(z)]\gamma_2^o, \tag{3}$$

where ε_x, ε_y and γ_{xy} are the inplane normal and shear strains. γ_{xz}, γ_{yz} are the transverse shear strains respectively. The superscripts x, y, z denote the partial derivative of the function with respect to x, y, z.

For a composite laminate of thickness h_k ($k = 1, 2, 3, \ldots$), and the ply-angle ϕ_k ($k = 1, 2, 3, \ldots$), the necessary expressions for computing the stiffness coefficients, available in the literature [9], are used. For the damping analysis, the complex moduli of an orthotropic material are defined, according to the elastic-viscoelastic correspondence principle, as follows:

$$E_1^* = E_1^R + \mathrm{i}E_1^I, \qquad E_2^* = E_2^R + \mathrm{i}E_2^I, \qquad E_3^* = E_3^R + \mathrm{i}E_3^I, \qquad G_{12}^* = G_{12}^R + \mathrm{i}G_{12}^I,$$
$$G_{23}^* = G_{23}^R + \mathrm{i}G_{23}^I, \qquad G_{13}^* = G_{13}^R + \mathrm{i}G_{13}^I. \tag{4}$$

Here, E^* and G^* are Young's modulus and shear modulus, respectively. The subscript 1 denotes longitudinal direction whereas 2 and 3 refer to the transverse directions, with respect to the fibres. The superscripts R and I denote the real and imaginary parts of the complex moduli. The material loss factors η_1, η_2, η_3 under tension-compression and $\eta_{12}, \eta_{23}, \eta_{13}$ under shear are defined as

$$\eta_1 = E_1^I/E_1^R, \qquad \eta_2 = E_2^I/E_2^R, \qquad \eta_3 = E_3^I/E_3^R, \qquad \eta_{12} = G_{12}^I/G_{12}^R,$$
$$\eta_{23} = G_{23}^I/G_{23}^R, \qquad \eta_{13} = G_{13}^I/G_{13}^R. \tag{5}$$

The strain energy of the laminate can be expressed in terms of the field variable $u, v, w, \theta_x, \theta_y$, and their derivatives. The kinetic energy includes the effect of in-plane and rotary inertia terms. Then, the governing equations are obtained by using Lagrange's equation of motion. These governing equations are solved using the finite element approach based on C^1 continuous elements developed recently based on the above theory.

For the plate analysis, an eight-node rectangular element is employed. The domain of the rectangular plate is divided into a number of elements. The element has eight degrees of freedom per corner node, namely $u, v, w, \partial w/\partial x, \partial w/\partial y, \partial^2 w/\partial x\,\partial y, \theta_x, \theta_y$, and four degrees of freedom per mid-side node, namely u, v, θ_x, θ_y. The shape functions employed to describe these field variables are the Hermite cubic function for transverse displacement w, and the Serendipity quadratic function for the in-plane displacements u, v and rotations θ_x, θ_y.

Similarly, for the beam problems, the three-node beam element used here is based on Hermite cubic functions for w and quadratic functions for rotation, θ_x, and linear function for u. Further, the element needs four nodal degrees of freedom $u, w, w_{,x}, \theta_x$ at both ends of the three-noded beam element, whereas the center node has one degree of freedom θ_x.

Using equation (3) and following the standard procedure given in reference [10], the finite element equations are derived as

$$[\mathbf{M}]\{\ddot{\boldsymbol{\delta}}\} + [[\mathbf{K}] + (1/2)[\mathbf{N}_1] + (1/3)[\mathbf{N}_2]]\{\boldsymbol{\delta}\} = \{\mathbf{0}\}, \tag{6}$$

where [**K**] is the linear stiffness matrix, and [$\mathbf{N}_1$] and [$\mathbf{N}_2$] are non-linear stiffness matrices. The stiffness matrices are of the complex form. $\{\boldsymbol{\delta}\}$ is the vector of the degrees of freedom associated with the displacement field in a finite element discretization. Substituting characteristics of the time function at the point of reversal of the motion

$$\{\ddot{\boldsymbol{\delta}}\}_{max} = -\lambda^*\{\boldsymbol{\delta}\}_{max} \tag{7}$$

in equation (6), will lead to the following non-linear algebraic equation of the form

$$[[\mathbf{K}] + (1/2)[\mathbf{N}_1] + (1/3)[\mathbf{N}_2]]\{\boldsymbol{\delta}\} - \lambda^*[\mathbf{M}]\{\boldsymbol{\delta}\} = \{\mathbf{0}\}. \tag{8}$$

The complex eigenvalues of the form $\lambda^* = (\lambda^R + \mathrm{i}\lambda^I) = (\omega^*)^2$ where $\omega^* = (\omega^R + \mathrm{i}\omega^I)$ are obtained for the above equation by using direct iteration technique, suitably modified for the eigenvalue problems based on the QR algorithm. The resonance frequencies ω and the system loss factors η are calculated from the eigenvalues, corresponding to different amplitudes of vibration level as:

$$\omega = \omega^R = (\lambda^R)^{1/2}, \qquad \eta = \lambda^I/\lambda^R. \tag{9}$$

3. NUMERICAL APPLICATION

Since the element is derived based on the field consistency approach, an exact numerical integration scheme is employed to evaluate all the strain energy terms. Also, there is no need to use the shear correction factor here, as the transverse strain is represented by a cosine function, which is of higher order in nature. Thus, the present development can be verified numerically by comparing the results based on different models, which are used for studying the thin and thick laminates. Such comparisons were made through linear and non-linear dynamic analysis, wherever possible, for the frequency values of composite laminates and excellent agreement was observed. For the sake of brevity, these results are not presented here. In this section, the system loss factors, as obtained from this work, will be discussed. Furthermore, based on progressive mesh refinement, 16 elements idealization and an 8×8 grid size are found to be adequate to model the laminated beams and plates, respectively, for the flexural/bending damping analysis. The damping factors obtained using linear analysis by 16 elements idealization for the beam and an 8×8 mesh size for the plate, are depicted in Table 1, and they are found to be in very good agreement with the available analytical/numerical solutions [5, 11].

The materials considered here are [12] as follows: GFRP(Glass/DX-210): $E_1^R = 37{\cdot}78$ GPa, $E_2^R = 10{\cdot}90$ GPa, $E_3^R = 10{\cdot}90$ GPa, $G_{12}^R = 4{\cdot}91$ GPa, $G_{23}^R = 4{\cdot}91$ GPa, $G_{13}^R = 4{\cdot}91$ GPa, $\nu_{12} = 0{\cdot}30$, $\eta_1 = 13{\cdot}8465 \times 10^{-4}$, $\eta_2 = \eta_3 = 8{\cdot}0373 \times 10^{-3}$, $\eta_{12} = \eta_{23} = \eta_{13} = 1{\cdot}09976 \times 10^{-2}$, $\rho = 1870$ kg/m^3; CFRP(HMS/DX-210): $E_1^R = 172{\cdot}70$ GPa, $E_2^R = 7{\cdot}20$ GPa, $E_3^R = 7{\cdot}20$ GPa, $G_{12}^R = 3{\cdot}76$ GPa, $G_{23}^R = 3{\cdot}76$ GPa, $G_{13}^R = 3{\cdot}76$ GPa, $\nu_{12} = 0{\cdot}30$, $\eta_1 = 7{\cdot}16197 \times 10^{-4}$, $\eta_2 = \eta_3 = 6{\cdot}71634 \times 10^{-3}$,

TABLE 1

Comparison of loss factors (η_L) of three-layered symmetric sandwich beams (thickness ratio of mid-to-outer layer = 1/7) and laminated plates (simply supported)

Beam[a] ($L/h = 30$)			Plate[b] (square)				
Young's modulus (mid layer) MPa	Present	Reference [11]	Lay-up	Present $L/h = 10$	Present 100	Reference [5] 10	Reference [5] 100
7·25	0·3204	0·32	0°	0·00497	0·00193	0·00489	0·00189
72·5	0·1134	0·1	$(0°/90°/0°/90°)_s$	0·00389	0·00191	0·00381	0·00189

[a] Mid-layer: Poisson's ratio (v) = 0·45; density (ρ) = 1200 kg/m^3; material loss factor (η) = 0·5.
Outer layers: Young's modulus = 45·54 GPa; v = 0·33; ρ = 2040 kg/m^3; η = 0·0.

[b] All layers are of equal thickness and same material, CFRP(HMS/DX-210).

TABLE 2

Loss factor ratio (η_{NL}/η_L) with vibration amplitudes (w/ρ_g) of simply supported laminated beams

Lay-up	w/ρ_g	$L/h = 10$	20	30	50	70	100	150	200
0°	0·1	0·99643	0·99715	0·99732	0·99741	0·99744	0·99745	0·99746	0·99746
	0·2	0·98596	0·98878	0·98944	0·98980	0·98991	0·98996	0·98999	0·99000
	0·4	0·94725	0·95770	0·96018	0·96154	0·96193	0·96214	0·96225	0·96229
	0·6	0·89221	0·91314	0·91814	0·92091	0·92170	0·92213	0·92236	0·92244
	0·8	0·83016	0·86247	0·87026	0·87458	0·87582	0·87648	0·87684	0·87697
	1	0·76852	0·81154	0·82201	0·82786	0·82953	0·83043	0·83091	0·83108
	Linear (η_L)	0·00223	0·00174	0·00164	0·00159	0·00158	0·00157	0·00157	0·00156
0°/90°/0°	0·1	0·99797	0·99864	0·99880	0·99889	0·99892	0·99893	0·99894	0·99894
	0·2	0·99199	0·99464	0·99527	0·99562	0·99572	0·99577	0·99580	0·99581
	0·4	0·96953	0·97954	0·98194	0·98327	0·98365	0·98385	0·98396	0·98400
	0·6	0·93653	0·95723	0·96224	0·96500	0·96579	0·96621	0·96644	0·96652
	0·8	0·89782	0·93082	0·93885	0·94330	0·94458	0·94526	0·94563	0·94576
	1	0·85759	0·90320	0·91436	0·92055	0·92233	0·92328	0·92379	0·92397
	Linear (η_L)	0·00227	0·00181	0·00171	0·00166	0·00165	0·00164	0·00164	0·00164
$(0°/90°/0°/90°)_s$	0·1	0·99799	0·99839	0·99848	0·99852	0·99853	0·99854	0·99854	0·99854
	0·2	0·99205	0·99364	0·99398	0·99415	0·99420	0·99423	0·99424	0·99425
	0·4	0·96979	0·97580	0·97706	0·97774	0·97792	0·97802	0·97808	0·97810
	0·6	0·93724	0·94957	0·95219	0·95358	0·95396	0·95417	0·95428	0·95432
	0·8	0·89930	0·91880	0·92295	0·92515	0·92577	0·92610	0·92628	0·92634
	1	0·86013	0·88680	0·89255	0·89560	0·89645	0·89691	0·89715	0·89724
	Linear (η_L)	0·00279	0·00245	0·00239	0·00235	0·00234	0·00234	0·00234	0·00233

TABLE 3

Loss factor ratio (η_{NL}/η_L) with vibration amplitudes (w/h) of simply supported laminated plates

Lay-up	w/h	$L/h = 10$	20	30	50	70	100	150	200
0°	0·1	0·94981	0·96421	0·96792	0·97022	0·97093	0·97133	0·97154	0·97162
	0·2	0·82698	0·87358	0·88626	0·89432	0·89683	0·89823	0·89899	0·89926
	0·4	0·58054	0·64338	0·70288	0·72275	0·72918	0·73280	0·73478	0·73549
	0·6	0·54473	0·50163	0·57722	0·60523	0·61468	0·62006	0·62304	0·62410
	Linear (η_L)	0·00497	0·00291	0·00237	0·00206	0·00197	0·00193	0·00190	0·00189
0°/90°/0°	0·1	0·95103	0·96456	0·96804	0·97020	0·97087	0·97124	0·97145	0·97152
	0·2	0·83024	0·87400	0·88581	0·89327	0·89559	0·89689	0·89760	0·89785
	0·4	0·56634	0·65387	0·68246	0·70131	0·70729	0·71065	0·71251	0·71317
	0·6	–	0·49917	0·52765	0·56016	0·56802	0·57248	0·57495	0·57583
	Linear (η_L)	0·00474	0·00283	0·00233	0·00205	0·00197	0·00192	0·00190	0·00189
(0°/90°/0°/90°)$_s$	0·1	0·95615	0·96662	0·96917	0·97068	0·97113	0·97137	0·97151	0·97156
	0·2	0·84650	0·88100	0·88972	0·89491	0·89646	0·89732	0·89779	0·89795
	0·4	0·60368	0·67105	0·69926	0·71141	0·71511	0·71716	0·71829	0·71869
	0·6	0·41930	0·52721	0·55206	0·54854	0·55427	0·55744	0·55917	0·55979
	0·8	–	0·43928	0·46709	0·45256	0·45930	0·46303	0·46507	0·46579
	1	–	–	0·35028	0·38943	0·43635	0·43842	0·40277	0·40369
	Linear (η_L)	0·00389	0·00250	0·00217	0·00199	0·00193	0·00191	0·00189	0·00189

$\eta_{12} = \eta_{23} = \eta_{13} = 1{\cdot}12204 \times 10^{-2}$, $\rho = 1566$ kg/m^3. ν_{12} and ρ are Poisson's ratio and mass density, respectively.

Numerical results are evaluated using eigenvalue formulation based on the QR algorithm. To solve the non-linear eigenvalue problem, an iterative procedure is used. The iteration starts from a corresponding initial mode shape obtained from linear analysis, with amplitude scaled up by a factor. This gives the initial value denoted by δ_i. Based on this initial mode shape, the non-linear stiffness matrices are formed, and an eigenvalue and its corresponding vector are evaluated. This eigenvector is then scaled up again and the iteration continues until the frequency/damping factor and the eigenvector obtained from the subsequent two iterations satisfy the required convergence criteria suggested by Bergan and Clough [13] within the tolerance of 0·01%.

The numerical experiments are conducted for analyzing the non-linear damping behavior of immovable, simply supported cross-ply laminates. The beam assumed here is made of glass fibre reinforced plastic material (GFRP), whereas carbon fibre reinforced plastic material (CFRP) is considered for the plate analysis. The results, concerning the first resonant mode, are presented in Tables 2 and 3 for beams and plates, respectively.

It is evident from these tables that, in general, a decrease in the system loss factor ratio (η_{NL}/η_L; η_L, η_{NL} are the system loss factors obtained from linear and non-linear analysis) is seen with the increase in the amplitude of vibration (w/ρ_g or w/h; ρ_g is the radius of gyration of the beam) of the laminates. This effect is increased when the aspect ratio is smaller. Also, it can be seen that the rate of decrease of the damping ratio is lower, with respect to the amplitudes, with an increase in the aspect ratio. This type of trend in the damping behavior arises from a change in the shear energy due to shear of the laminates, and it depends not only on the aspect ratio, but also on the level of vibration amplitudes. It is also inferred from these tables that the value of the loss factor ratio changes with the number of layers in the laminates, as highlighted in reference [5]. However, for the laminated plates with a smaller number of layers, the results could not be obtained for cases with higher amplitudes of vibration due to the convergence problem. It is hoped that this study will be useful for the designers/engineers while designing the composite laminate for the flexural response under dynamic situations.

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